AIAA JOURNAL
Vol. 40, No. 4, April 2002

Higher-Order Eigensensitivity Analysis of a Defective Matrix

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Under the assumption that all of the eigenvalues of the eigenproblem for the first-order perturbation coefficients of the eigenvalues are simple and nonzero, a direct method is given for calculating the first- to third-order perturbation coefficients of the eigenvalues and the first- to second-order perturbation coefficients of the eigenvectors of a defective matrix. The method is extended to the case where one of the first-order perturbation coefficients of the eigenvalues associated with the lowest-order Jordan blocks is zero. Numerical examples show the validity of the method.

Nomenclature

$\boldsymbol{A}$	=	concerned $n \times n$ matrix
$\boldsymbol{B}$	=	$n \times n$ perturbation matrix
$d_1, \ldots, d_r$	=	different orders of Jordan blocks of $\boldsymbol{A}$ associated with eigenvalue λ_0 (It is assumed that $d_1 < \cdots < d_r$ when $r > 1$.)
$\boldsymbol{W}$	=	matrix whose columns are all of the eigenvectors of the perturbed matrix $\boldsymbol{A} + \varepsilon \boldsymbol{B}$ associated with d_tth-order blocks of $\boldsymbol{A}$ corresponding to λ_0
$\boldsymbol{W}^{(k)}$	=	kth-order perturbation coefficient matrix of $\boldsymbol{W}$ $(k \geq 1)$
ε	=	small positive perturbation parameter
Λ	=	diagonal matrix whose diagonal elements are all of the eigenvalues of $\boldsymbol{A} + \varepsilon \boldsymbol{B}$ associated with d_tth-order blocks of $\boldsymbol{A}$ corresponding to λ_0
$\Lambda^{(k)}$	=	kth-order perturbation coefficient matrix of Λ $(k \geq 1)$
λ_0	=	concerned eigenvalue of $\boldsymbol{A}$

Superscripts

H	=	transpose and complex conjugate of a matrix
$-$	=	complex conjugate of a variable

Introduction

THERE are many important applications of eigensensitivity analysis to dynamical analysis, identification and modification of engineering structures, vibration control, and optimization. The sensitivity analysis of simple eigenvalues is easy. The sensitivity analysis of repeated eigenvalues is much more difficult, and most of the work is concentrated on the nondefective system.[1–3] However many defective systems can be encountered in practice.[4] There are only a few works on eigensensitivity analysis of a defective system because of its difficulties. Luongo[5] and Zhang and Zhang[6] gave the methods for calculating the first-order perturbation coefficients of the eigenvectors and some higher-order perturbation coefficients of the eigenvalues of a defective matrix. It is well known that the eigenvalue problem of a defective matrix is very sensitive to perturbations.[7] Therefore there are requests in practical engineering to do some higher-order eigensensitivity analysis of a defective matrix. Besides, the methods in Refs. 5 and 6 cannot be applied to the case where some of the first-order perturbation coefficients of the eigenvalues associated with Jordan blocks with order greater than one are zero. The purpose of this paper is to give the method for calculating the higher-order perturbation coefficients of eigenvalues and eigenvectors of a defective matrix and to extend the method to the case where some of the first-order perturbation coefficients of the eigenvalues are zero.

Received 28 March 2001; revision received 28 September 2001; accepted for publication 5 November 2001.

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Formulation of the Problem

Suppose that in the Jordan canonical form of $\boldsymbol{A}$ associated with λ_0 there are $s(k)$ blocks with order $d(k)(k = 1, \ldots, r)$. Let $\boldsymbol{v}_j^{(k)}$ and $\boldsymbol{u}_j^{(k,l)}$ be, respectively, the left eigenvector and the lth-order principal vector associated with the jth block of order $d(k)[l = 1, \ldots, d(k); j = 1, \ldots, s(k); k = 1, \ldots, r]$. Define

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} - \lambda_0 \boldsymbol{I}, \qquad \boldsymbol{V}^{(k)} = \left[\boldsymbol{v}_1^{(k)}, \ldots, \boldsymbol{v}_{s(k)}^{(k)}\right], \qquad \boldsymbol{U}^{(k,0)} = 0$$

$$\boldsymbol{U}^{(k,l)} = \left[\boldsymbol{u}_1^{(k,l)}, \ldots, \boldsymbol{u}_{s(k)}^{(k,l)}\right], \qquad l = 1, \ldots, d(k), \quad k = 1, \ldots, r$$

Then $\boldsymbol{V}^{(k)}$ and $\boldsymbol{U}^{(k,l)}$ can be made satisfying

$$\tilde{\boldsymbol{A}} \boldsymbol{U}^{(k,l)} = \boldsymbol{U}^{(k,l-1)}$$

$$\boldsymbol{V}^{(j)H} \boldsymbol{U}^{(k,l)} = \begin{cases} \boldsymbol{I}_{s(k)}, & \text{when} \quad j = k, \quad l = d(k) \\ 0, & \text{otherwise} \end{cases}$$

$$l = 1, \ldots, d(k), \quad j, k = 1, \ldots, r$$

Now we investigate the variations of the eigenvalues and eigenvectors associated with the $s(t)$ blocks of order $d(t)$(according to the increasing orders $t = 1, \ldots, r$) when $\boldsymbol{A}$ is perturbed by $\varepsilon \boldsymbol{B}$. In this paper it is assumed that the perturbed problem is nondefective. Thus the perturbed problem can be expressed as

$$(\boldsymbol{A} + \varepsilon \boldsymbol{B}) \boldsymbol{W} = \boldsymbol{W} \Lambda \tag{1}$$

where the columns of $\boldsymbol{W} = [\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{s(t)}]$ are linearly independent eigenvectors of $\boldsymbol{A} + \varepsilon \boldsymbol{B}$ and the diagonal elements of $\Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_{s(t)}]$ are the corresponding eigenvalues. Define the known $s(j) \times s(k)$ matrices

$$\boldsymbol{Q}^{(j,k)} = \boldsymbol{V}^{(j)H} \boldsymbol{B} \boldsymbol{U}^{(k,1)}, \qquad j, k = 1, \ldots, r$$

In this paper it is assumed first that

$$\Delta_k \stackrel{\text{def}}{=} \begin{vmatrix} \boldsymbol{Q}^{(k,k)} & \cdots & \boldsymbol{Q}^{(k,r)} \\ \cdots & \cdots & \cdots \\ \boldsymbol{Q}^{(r,k)} & \cdots & \boldsymbol{Q}^{(r,r)} \end{vmatrix} \neq 0, \qquad k = 1, \ldots, r \tag{2}$$

Expand $\boldsymbol{W}$ and Λ in the Puiseux series of ε:

$$\boldsymbol{W} = \sum_{k=0}^{\infty} \boldsymbol{W}^{(k)} \eta^k, \qquad \Lambda = \sum_{k=0}^{\infty} \Lambda^{(k)} \eta^k \tag{3}$$

where

$$\eta = \varepsilon^{1/\nu}, \qquad \nu = d(t), \qquad \Lambda^{(0)} = \lambda_0 \boldsymbol{I}_{s(t)}$$

$$\boldsymbol{W}^{(k)} = \left[\boldsymbol{w}_1^{(k)}, \ldots, \boldsymbol{w}_{s(t)}^{(k)}\right]$$

$$\Lambda^{(k)} = \operatorname{diag}\left[\lambda_1^{(k)}, \ldots, \lambda_{s(t)}^{(k)}\right], \qquad k = 0, 1, \ldots$$

The columns of $W^{(0)}$ are a set of linearly independent differentiable eigenvectors of the unperturbed problem. The main purpose of this paper is to calculate $W^{(0)}$, $W^{(k)}$, and $\Lambda^{(k)}$, $(k = 1, 2, \ldots)$. In this paper it is assumed that each column of $W^{(0)}$ is so normalized that $w^{(0)}_{e(j),j}$, the first among the components with largest absolute value in column $w^{(0)}_j$, is 1:

$$w^{(0)}_{e(j),j} = 1, \qquad j = 1, \ldots, s(t) \tag{4a}$$

Therefore if it is stipulated that w_j, the column of W, is so normalized that its corresponding component is 1:

$$w_{e(j),j} = 1, \qquad j = 1, \ldots, s(t) \tag{4b}$$

then when $k \geq 1$, $w^{(k)}_j$, the corresponding column of $W^{(k)}$, must satisfy the condition

$$w^{(k)}_{e(j),j} = 0, \qquad j = 1, \ldots, s(t), \qquad k = 1, 2, \ldots \tag{4c}$$

Substituting Eqs. (3) into Eq. (1) and comparing the coefficients of the powers of η, we obtain

$$\tilde{A}W^{(j)} = \sum_{k=1}^{j} W^{(j-k)}\Lambda^{(k)}, \qquad j = 0, \ldots, \nu - 1 \tag{5a}$$

$$\tilde{A}W^{(j)} = \sum_{k=1}^{j} W^{(j-k)}\Lambda^{(k)} - BW^{(j-\nu)}, \qquad j = \nu, \nu + 1, \ldots \tag{5b}$$

The equation for $W^{(\nu)}$ is[6]

$$\tilde{A}W^{(\nu)} = \sum_{k=1}^{t-1} U^{[k,d(k)]}C^{[k,\nu-d(k)]}\Lambda^{(1)^{d(k)}} + \sum_{k=t}^{r} U^{(k,\nu)}C^{(k,0)}\Lambda^{(1)^{\nu}} - BW^{(0)} + G^{(0)} \overset{\text{def}}{=} R^{(0)} + G^{(0)} \tag{6}$$

The expressions of $G^{(l)}$ and $\tilde{G}^{(l)}$, which we have used and will use in this paper, are defined by

$$G^{(l)} = \sum_{k=1}^{t-1}\sum_{j=1}^{d(k)-1} U^{(k,j)}E^{[d(k)+l,\nu+l,k,j]} + \sum_{k=t}^{r}\sum_{j=1}^{\nu} U^{(k,j)}E^{(\nu+l,\nu+l,k,j)}$$

$$\tilde{G}^{(l)} = \sum_{k=1}^{t-1}\sum_{j=1}^{d(k)-1} U^{(k,j+1)}E^{[d(k)+l,\nu+l,k,j]} + \sum_{k=t}^{r}\sum_{j=1}^{\nu-1} U^{(k,j+1)}E^{(\nu+l,\nu+l,k,j)}$$

where

$$E^{(m,l,k,j)} \overset{\text{def}}{=} \sum_{p=j}^{m} C^{(k,l-p)}, \qquad \sum_{\substack{h_1,\ldots,h_j \geq 1 \\ h_1+\cdots+h_j = p}} \prod_{q=1}^{j} \Lambda^{(h_q)}$$

and $C^{(k,j)} = [c^{(k,j)}_{\text{ml}}] = [c^{(k,j)}_1, \ldots, c^{(k,j)}_{s(t)}]$ is a $s(k) \times s(t)$ coefficient matrix to be determined $(k = 1, \ldots, r;\ j = 0, 1, \ldots)$. On the right-hand side of Eq. (6), all of the terms except $R^{(0)}$ are in $\Re(\tilde{A})$, the ranger of $\tilde{A}$. From the solvability condition of Eq. (6), $V^{(j)H}R^{(0)} = 0$ $(j = 1, \ldots, r)$, we obtain[6]

$$\left[Q^{(t,t)} + \sum_{k=t+1}^{r} Q^{(t,k)}Q^{(k)}\right]C^{(t,0)} = C^{(t,0)}\Lambda^{(1)^{\nu}} \tag{7}$$

where the known quantities $Q^{(t+1)}, \ldots, Q^{(r)}$ are defined by

$$\begin{bmatrix} Q^{(t+1)} \\ \vdots \\ Q^{(r)} \end{bmatrix} = -\begin{bmatrix} Q^{(t+1,t+1)} & \cdots & Q^{(t+1,r)} \\ \cdots & \cdots & \cdots \\ Q^{(r,t+1)} & \cdots & Q^{(r,r)} \end{bmatrix}^{-1} \begin{bmatrix} Q^{(t+1,t)} \\ \vdots \\ Q^{(r,t)} \end{bmatrix} \overset{\text{def}}{=} -Q\begin{bmatrix} Q^{(t+1,t)} \\ \vdots \\ Q^{(r,t)} \end{bmatrix} \tag{8}$$

Equation (7) is an eigenvalue problem with the columns of $C^{(t,0)}$ as its eigenvectors and the diagonal elements of $\Lambda^{(1)^{\nu}}$ as its corresponding eigenvalues. In this paper it is assumed that all of the eigenvalues of problem (7) are simple.

Case 1: $\Delta_k \neq 0$ $(k = 1, \ldots, r)$

In this case for any t all of the eigenvalues of problem (7) are nonzero, that is, $\Lambda^{(1)}$ is nonsingular (see the appendix in Ref. 6). Take arbitrarily $s(t)$ linearly independent eigenvectors of problem (7) and construct an $s(t) \times s(t)$ matrix $\tilde{C}^{(t,0)}$ with these eigenvectors as its columns. Define

$$\tilde{W}^{(0)} = \left[\tilde{w}^{(0)}_1, \ldots, \tilde{w}^{(0)}_{s(t)}\right] \overset{\text{def}}{=} \left[U^{(t,1)} + \sum_{k=t+1}^{r} U^{(k,1)}Q^{(k)}\right]\tilde{C}^{(t,0)}$$

If the first among the components with largest absolute values in column $\tilde{w}^{(0)}_j$ is $\tilde{w}^{(0)}_{e(j),j}$ $[j = 1, \ldots, s(t)]$, then the columns of $C^{(t,0)}$ and $W^{(0)}$ are determined by

$$w^{(0)}_j = \tilde{w}^{(0)}_j \Big/ \tilde{w}^{(0)}_{e(j),j}, \qquad c^{(t,0)}_j = \tilde{c}^{(t,0)}_j \Big/ \tilde{w}^{(0)}_{e(j),j}, \qquad j = 1, \ldots, s(t)$$

Thus $C^{(t,0)}$, $C^{(k,0)} = Q^{(k)}C^{(t,0)}$ $(k = t+1, \ldots, r)$ and $W^{(0)}$ are determined. Then $C^{[j,\nu-d(j)]}$ $(j = 1, \ldots, t-1)$ can be determined by

$$C^{[j,\nu-d(j)]} = \left[\sum_{k=t}^{r} Q^{(j,k)}C^{(k,0)}\right]\Lambda^{(1)^{-d(j)}}, \qquad j = 1, \ldots, t-1 \tag{9}$$

Then $R^{(0)}$ defined in Eq. (6) can be determined and $\tilde{W}^{(\nu)} = \tilde{A}^{(1)}R^{(0)}$ can be calculated, where $\tilde{A}^{(1)}$ is any generalized {1}-inverse of $\tilde{A}$. The general solution of Eq. (6) can be expressed as

$$W^{(\nu)} = \tilde{W}^{(\nu)} + \tilde{G}^{(0)} + \sum_{k=1}^{r} U^{(k,1)}C^{(k,\nu)}$$

Substituting all of the preceding results into the equation for $W^{(\nu+1)}$ in Eqs. (5b), we obtain

$$\tilde{A}W^{(\nu+1)} = \tilde{W}^{(\nu)}\Lambda^{(1)} + \sum_{k=1}^{t-1} U^{[k,d(k)]}D^{(1,k,k)} + \sum_{k=t}^{r} U^{(k,\nu)}D^{(1,k,t)} - BW^{(1)} + G^{(1)} \overset{\text{def}}{=} R^{(1)} + G^{(1)} \tag{10}$$

where $D^{(1,j,k)}$ is defined by

$$D^{(1,j,k)} = C^{[j,\nu+1-d(k)]}\Lambda^{(1)^{d(k)}} + d(k)C^{[j,\nu-d(k)]}\Lambda^{(1)^{d(k)-1}}\Lambda^{(2)}$$

All of the terms except $R^{(1)}$ on the right-hand side of Eq. (10) are in $\Re(\tilde{A})$. When $t > 1$ and $d(t) = d(t-1) + 1$, $C^{(t-1,1)}$ has been calculated by Eq. (9). Therefore the following defined quantities are known:

$$\overset{*}{W}^{(1)} = \begin{cases} \tilde{W}^{(1)}, & \nu = 1 \\ \sum_{k=t}^{r} U^{(k,2)}C^{(k,0)}\Lambda^{(1)}, & t = 1, \quad \nu > 1 \quad \text{or} \quad t > 1, \quad \nu > d(t-1) + 1 \\ \sum_{k=t}^{r} U^{(k,2)}C^{(k,0)}\Lambda^{(1)} + U^{(t-1,1)}C^{(t-1,1)}, & \text{otherwise} \end{cases}$$

$$f^{(1,j)} = V^{(j)^H}\big[\tilde{W}^{(\nu)}\Lambda^{(1)} - B\overset{*}{W}{}^{(1)}\big], \qquad j = 1, \dots, r$$

From the solvability condition of Eq. (10), $V^{(j)H}R^{(1)} = 0\ (j = 1, \dots, r)$, it follows that

$$\sum_{k=t}^{r} Q^{(j,k)}C^{(k,1)} = D^{(1,j,j)} + f^{(1,j)}, \qquad j = 1, \dots, t \quad (11a)$$

$$C^{(k,1)} = Q^{(k)}C^{(t,1)} + g^{(1,k)}, \qquad k = t+1, \dots, r \quad (11b)$$

where $g^{(1,k)}$ $(k = t+1, \dots, r)$ are defined by

$$\begin{bmatrix} g^{(1,t+1)} \\ \vdots \\ g^{(1,r)} \end{bmatrix} = Q \begin{bmatrix} f^{(1,t+1)} \\ \vdots \\ f^{(1,r)} \end{bmatrix}$$

and $Q^{(k)}$ $(k = t+1, \dots, r)$ and Q are defined by Eq. (8). Substituting Eqs. (11b) into the tth equation in Eqs. (11a) and using Eqs. (7), we obtain

$$\Lambda^{(1)^{\nu}}\tilde{C}^{(t,1)} - \tilde{C}^{(t,1)}\Lambda^{(1)^{\nu}} = \nu\Lambda^{(1)^{\nu-1}}\Lambda^{(2)} + f^{(1)} \qquad (12)$$

where the $s(t) \times s(t)$ matrix $\tilde{C}^{(t,1)} = [\tilde{c}^{(t,1)}_{\mathrm{ml}}]$ and the known matrix $f^{(1)} = [f^{(1)}_{\mathrm{ml}}]$ are defined by

$$\tilde{C}^{(t,1)} = C^{(t,0)^{-1}}C^{(t,1)}, \qquad f^{(1)} = C^{(t,0)^{-1}}\left[f^{(1,t)} - \sum_{k=t+1}^{r} Q^{(t,k)}g^{(1,k)}\right]$$

Comparing the diagonal elements of the matrices on both sides of Eq. (12), we obtain $\Lambda^{(2)}$:

$$\lambda^{(2)}_{j} = -\frac{f^{(1)}_{jj}}{\nu\lambda_j^{(1)^{\nu-1}}}, \qquad j = 1, \dots, s(t)$$

Comparing the nondiagonal elements of the matrices on both sides of Eq. (12), we obtain the nondiagonal elements of $\tilde{C}^{(t,1)}$:

$$\tilde{c}^{(t,1)}_{jk} = \frac{f^{(1)}_{jk}}{\lambda_j^{(1)^{\nu}} - \lambda_k^{(1)^{\nu}}}, \qquad j \neq k, \qquad j, k = 1, \dots, s(t)$$

By using Eqs. (11b), the expression for $W^{(1)}$ can be rewritten as

$$W^{(1)} = W^{(0)}\tilde{C}^{(t,1)} + g^{(1)} \qquad (13)$$

where the known $n \times s(t)$ matrix $g^{(1)} = [g^{(1)}_{\mathrm{ml}}]$ is defined by

$$g^{(1)} = \overset{*}{W}{}^{(1)} + \sum_{k=t+1}^{r} U^{(k,1)}g^{(1,k)}$$

By using Eqs. (4a) and (4c), the diagonal elements of $\tilde{C}^{(t,1)}$ can be obtained from Eqs. (13):

$$\tilde{c}^{(t,1)}_{jj} = -\left[\sum_{\substack{k=1 \\ k\neq j}}^{s(t)} w^{(0)}_{e(j),k}\tilde{c}^{(t,1)}_{kj} + g^{(1)}_{e(j),j}\right], \qquad j = 1, \dots, s(t)$$

Now $\tilde{C}^{(t,1)}$ is completely determined. Thus, $C^{(t,1)} = C^{(t,0)}\tilde{C}^{(t,1)}$, $C^{(k,1)} = Q^{(k)}C^{(t,1)} + g^{(1,k)}$ $(k = t+1, \dots, r)$, and $W^{(1)}$ defined by Eq. (13) can be calculated in turns. Using Eqs. (11a), we obtain

$$C^{[j,\nu+1-d(j)]} = \left[\sum_{k=t}^{r} Q^{(j,k)}C^{(k,1)} - d(j)C^{[j,\nu-d(j)]} \times \Lambda^{(1)^{d(j)-1}}\Lambda^{(2)} - f^{(1,j)}\right]\Lambda^{(1)^{-d(j)}}, \qquad j = 1, \dots, t-1$$

Then $R^{(1)}$ defined in Eq. (10) can be determined, and $\tilde{W}^{(\nu+1)} = \tilde{A}^{(1)}R^{(1)}$ can be calculated. The general solution of Eq. (10) can be expressed as

$$W^{(\nu+1)} = \tilde{W}^{(\nu+1)} + \tilde{G}^{(1)} + \sum_{k=1}^{r} U^{(k,1)}C^{(k,\nu+1)}$$

Substituting all of the preceding results into the equation for $W^{(\nu+2)}$ in Eqs. (5b), we obtain

$$\tilde{A}W^{(\nu+2)} = \sum_{k=1}^{t-1} U^{[k,d(k)]}D^{(2,k,k)} + \sum_{k=t}^{r} U^{(k,1)}D^{(2,k,t)} + \tilde{W}^{(\nu+1)}\Lambda^{(1)} + \tilde{W}^{(\nu)}\Lambda^{(2)} - BW^{(2)} + G^{(2)} \overset{\mathrm{def}}{=} R^{(2)} + G^{(2)} \qquad (14)$$

where $D^{(2,j,k)}$ is defined by

$$D^{(2,j,k)} = C^{[j,\nu+2-d(k)]}\Lambda^{(1)^{d(k)}} + d(k)C^{[j,\nu+1-d(k)]}\Lambda^{(1)^{d(k)-1}}\Lambda^{(2)} + C^{[j,\nu-d(k)]} \times \left\{d(k)\Lambda^{(1)^{d(k)-1}}\Lambda^{(3)} + \frac{d(k)[d(k)-1]}{2}\Lambda^{(1)^{d(k)-2}}\Lambda^{(2)^2}\right\}$$

All of the terms except $R^{(2)}$ on the right-hand side of Eq. (14) are in $\Re(\tilde{A})$. Define the following known quantities:

$$T = \begin{cases} \tilde{W}^{(2)}, & \nu = 1 \\ \tilde{W}^{(2)} + \sum_{k=t}^{r} U^{(k,2)}\big[C^{(k,1)}\Lambda^{(1)} + C^{(k,0)}\Lambda^{(2)}\big], & \nu = 2 \\ \sum_{k=t}^{r}\big\{U^{(k,3)}C^{(k,0)}\Lambda^{(1)^2} + U^{(k,2)}\big[C^{(k,1)}\Lambda^{(1)} + C^{(k,0)}\Lambda^{(2)}\big]\big\}, & \nu > 2 \end{cases}$$

$$\overset{*}{W}{}^{(2)} = \begin{cases} T + \sum_{k=t-2}^{t-1} U^{(k,1)}C^{(k,2)}, & t > 2, \quad \nu = d(t-1)+1 = d(t-2)+2 \\ T + U^{(t-1,1)}C^{(t-1,2)}, & t = 2, \quad \nu = d(t-1)+1 \quad \text{or} \quad t > 2, \quad \nu = d(t-1)+1 > d(t-2)+2 \\ T, & \text{otherwise} \end{cases}$$

$$f^{(2,j)} = V^{(j)^H}\big[\tilde{W}^{(\nu+1)}\Lambda^{(1)} + \tilde{W}^{(\nu)}\Lambda^{(2)} - B\overset{*}{W}{}^{(2)}\big], \qquad j = 1, \dots, r$$

From the solvability condition of Eq. (14), $V^{(j)H}R^{(2)} = 0\ (j = 1, \dots, r)$, it follows that

$$\sum_{k=t}^{r} Q^{(j,k)}C^{(k,2)} = D^{(2,j,j)}f^{(2,j)}, \qquad j = 1, \dots, t \quad (15a)$$

$$C^{(k,2)} = Q^{(k)}C^{(t,2)} + g^{(2,k)}, \qquad k = t+1, \dots, r \quad (15b)$$

where $\boldsymbol{g}^{(2,k)} (k = t+1, \ldots, r)$ are defined by

$$\begin{bmatrix} \boldsymbol{g}^{(2,t+1)} \\ \vdots \\ \boldsymbol{g}^{(2,r)} \end{bmatrix} = \boldsymbol{Q} \begin{bmatrix} \boldsymbol{f}^{(2,t+1)} \\ \vdots \\ \boldsymbol{f}^{(2,r)} \end{bmatrix}$$

Substituting Eqs. (15b) into the tth equation in Eqs. (15a), we obtain

$$\Lambda^{(1)^\nu} \tilde{\boldsymbol{C}}^{(t,2)} - \tilde{\boldsymbol{C}}^{(t,2)} \Lambda^{(1)^\nu} = \nu \Lambda^{(1)^{\nu-1}} \Lambda^{(3)} + \boldsymbol{f}^{(2)} \tag{16}$$

where $\tilde{\boldsymbol{C}}^{(t,2)}$ and the known matrix $\boldsymbol{f}^{(2)} = [f^{(2)}_{\mathrm{ml}}]$ are defined by

$$\tilde{\boldsymbol{C}}^{(t,2)} = \left[\tilde{c}^{(t,2)}_{\mathrm{ml}}\right] = \boldsymbol{C}^{(t,0)^{-1}} \boldsymbol{C}^{(t,2)}$$

$$\boldsymbol{f}^{(2)} = \boldsymbol{C}^{(t,0)^{-1}} \left[\boldsymbol{f}^{(2,t)} - \sum_{k=t+1}^{r} \boldsymbol{Q}^{(t,k)} \boldsymbol{g}^{(2,k)} \right]$$

$$+ \nu \tilde{\boldsymbol{C}}^{(t,1)} \Lambda^{(1)^{\nu-1}} \Lambda^{(2)} + \frac{\nu(\nu-1)}{2} \Lambda^{(1)^{\nu-2}} \Lambda^{(2)^2}$$

Comparing the diagonal elements of the matrices on both sides of Eq. (16), we obtain $\Lambda^{(3)}$:

$$\lambda^{(3)}_j = -\frac{f^{(2)}_{jj}}{\nu \lambda^{(1)^{\nu-1}}_j}, \qquad j = 1, \ldots, s(t)$$

Comparing the nondiagonal elements of the matrices on both sides of Eq. (16), we obtain the nondiagonal elements of $\tilde{\boldsymbol{C}}^{(t,2)}$:

$$\tilde{c}^{(t,2)}_{jk} = \frac{f^{(2)}_{jk}}{\lambda^{(1)^\nu}_j - \lambda^{(1)^\nu}_k}, \qquad j \neq k, \qquad j, k = 1, \ldots, s(t)$$

By using Eqs. (15b), the expression for $\boldsymbol{W}^{(2)}$ can be rewritten as

$$\boldsymbol{W}^{(2)} = \boldsymbol{W}^{(0)} \tilde{\boldsymbol{C}}^{(t,2)} + \boldsymbol{g}^{(2)} \tag{17}$$

where the known matrix $\boldsymbol{g}^{(2)} = [g^{(2)}_{\mathrm{ml}}]$ is defined by

$$\boldsymbol{g}^{(2)} = \overset{*}{\boldsymbol{W}}{}^{(2)} + \sum_{k=t+1}^{r} \boldsymbol{U}^{(k,1)} \boldsymbol{g}^{(2,k)}$$

By using Eqs. (4a) and (4c), the diagonal elements of $\tilde{\boldsymbol{C}}^{(t,2)}$ can be obtained from Eq. (17):

$$\tilde{c}^{(t,2)}_{jj} = -\left[\sum_{\substack{k=1 \\ k \neq j}}^{s(t)} w^{(0)}_{e(j),k} \tilde{c}^{(t,2)}_{kj} + g^{(2)}_{e(j),j} \right], \qquad j = 1, \ldots, s(t)$$

Thus $\tilde{\boldsymbol{C}}^{(t,2)}$ is completely determined. Then $\boldsymbol{C}^{(t,2)} = \boldsymbol{C}^{(t,0)} \tilde{\boldsymbol{C}}^{(t,2)}$, $\boldsymbol{C}^{(k,2)} = \boldsymbol{Q}^{(k)} \boldsymbol{C}^{(t,2)} + \boldsymbol{g}^{(2,k)}$ $(k = t+1, \ldots, r)$, and $\boldsymbol{W}^{(2)}$ defined by Eq. (17) can be calculated in turns. Using Eqs. (15a), we obtain

$$\boldsymbol{C}^{[j,\nu+2-d(j)]} = \left(\sum_{k=t}^{r} \boldsymbol{Q}^{(j,k)} \boldsymbol{C}^{(k,2)} - d(j) \boldsymbol{C}^{[j,\nu+1-d(j)]} \Lambda^{(1)^{d(j)-1}} \Lambda^{(2)} \right.$$

$$- \boldsymbol{C}^{[j,\nu-d(j)]} \left\{ d(j) \Lambda^{(1)^{d(j)-1}} \Lambda^{(3)} + \frac{d(j)[d(j)-1]}{2} \right.$$

$$\left. \left. \times \Lambda^{(1)^{d(j)-2}} \Lambda^{(2)^2} \right\} - \boldsymbol{f}^{(2,j)} \right) \Lambda^{(1)^{-d(j)}}, \qquad j = 1, \ldots, t-1$$

Then $\boldsymbol{R}^{(2)}$ defined in Eq. (14) can be determined, and $\tilde{\boldsymbol{W}}^{(\nu+2)} = \tilde{\boldsymbol{A}}^{(1)} \boldsymbol{R}^{(2)}$ can be calculated. The general solution of Eq. (14) can be expressed as

$$\boldsymbol{W}^{(\nu+2)} = \tilde{\boldsymbol{W}}^{(\nu+2)} + \tilde{\boldsymbol{G}}^{(2)} + \sum_{k=1}^{r} \boldsymbol{U}^{(k,1)} \boldsymbol{C}^{(k,\nu+2)} \tag{18}$$

Substituting all of the preceding results into the equation for $\boldsymbol{W}^{(\nu+3)}$ in Eqs. (5b) and using the solvability condition of the resultant equation, we can determine $\Lambda^{(4)}$ and $\boldsymbol{W}^{(3)}$. In a similar way, the higher-order perturbation coefficients of the eigenvalues and eigenvectors can be calculated from the higher-order perturbation equations.

Case 2: $\Delta_1 = 0$, $\Delta_k \neq 0$ $(k = 2, \ldots, r)$

In this case, when $t > 1$ or when $t = 1$ and $d(1) = 1$, the preceding method in case 1 can be used to calculate the higher-order perturbation coefficients of the eigenvalues and eigenvectors associated with the $d(t)$th order Jordan blocks, but the preceding method cannot be used to calculate the higher-order perturbation coefficients of the eigenvalues and eigenvectors associated with $d(1)$th-order Jordan blocks if $d(1) > 1$ because some of the first-order perturbation coefficients of the eigenvalues is zero. In this case, therefore, we only have to consider how to calculate the perturbation coefficients associated with the $d(1)$th-order Jordan blocks with order $\nu = d(1) > 1$.

Under the assumption that the eigenvalues of problem (7) are all simple, problem (7) only has one zero eigenvalue. For certain we assume that $\lambda^{(1)}_1 = 0$. Thus, $\lambda^{(1)}_j \neq 0$ $(j \neq 1)$. In this case $\tilde{\boldsymbol{C}}^{(t,1)}$ and $\lambda^{(2)}_j$ $[j = 2, \ldots, s(1)]$, the other diagonal elements of $\Lambda^{(2)}$ can be calculated from Eq. (12) by use of the same way in case 1, but $\lambda^{(2)}_1$ cannot be determined from Eq. (12). Define the following known quantities:

$$\hat{\boldsymbol{W}}^{(2)} = \begin{cases} \tilde{\boldsymbol{W}}^{(2)} + \displaystyle\sum_{k=1}^{r} \boldsymbol{U}^{(k,2)} \boldsymbol{C}^{(k,1)} \Lambda^{(1)}, & \nu = 2 \\ \displaystyle\sum_{k=1}^{r} \left[\boldsymbol{U}^{(k,3)} \boldsymbol{C}^{(k,0)} \Lambda^{(1)^2} + \boldsymbol{U}^{(k,2)} \boldsymbol{C}^{(k,1)} \Lambda^{(1)} \right], & \nu > 2 \end{cases}$$

$$\hat{\boldsymbol{f}}^{(2,j)} = \boldsymbol{V}^{(j)^H} \left[\tilde{\boldsymbol{W}}^{(\nu+1)} \Lambda^{(1)} - \boldsymbol{B} \hat{\boldsymbol{W}}^{(2)} \right], \qquad j = 1, \ldots, r$$

$$\hat{\boldsymbol{h}}^{(j)} = \boldsymbol{V}^{(j)^H} \left[\tilde{\boldsymbol{W}}^{(\nu)} - \boldsymbol{B} \sum_{k=1}^{r} \boldsymbol{U}^{(k,2)} \boldsymbol{C}^{(k,0)} \right], \qquad j = 1, \ldots, r$$

To calculate $\lambda^{(2)}_1$, we rewrite Eq. (16) as

$$\Lambda^{(1)^\nu} \tilde{\boldsymbol{C}}^{(1,2)} - \tilde{\boldsymbol{C}}^{(1,2)} \Lambda^{(1)^\nu} = \nu \Lambda^{(1)^{\nu-1}} \Lambda^{(3)} + \frac{\nu(\nu-1)}{2} \Lambda^{(1)^{\nu-2}} \Lambda^{(2)^2}$$

$$+ \nu \tilde{\boldsymbol{C}}^{(1,1)} \Lambda^{(1)^{\nu-1}} \Lambda^{(2)} + \boldsymbol{h} \Lambda^{(2)} + \hat{\boldsymbol{f}}^{(2)} \tag{19}$$

where

$$\hat{\boldsymbol{f}}^{(2)} = \left[\hat{f}^{(2)}_{\mathrm{ml}}\right] \overset{\mathrm{def}}{=} \boldsymbol{C}^{(1,0)^{-1}} \left[\hat{\boldsymbol{f}}^{(2,1)} - \sum_{k=2}^{r} \boldsymbol{Q}^{(1,k)} \hat{\boldsymbol{g}}^{(2,k)} \right]$$

$$\boldsymbol{h} = (h_{\mathrm{ml}}) \overset{\mathrm{def}}{=} \boldsymbol{C}^{(1,0)^{-1}} \left[\hat{\boldsymbol{h}}^{(1)} - \sum_{k=2}^{r} \boldsymbol{Q}^{(1,k)} \boldsymbol{h}^{(k)} \right]$$

and $\hat{\boldsymbol{g}}^{(2,k)}$ and $\boldsymbol{h}^{(k)}$ $(k = 2, \ldots, r)$ are defined by

$$\begin{bmatrix} \hat{\boldsymbol{g}}^{(2,2)} & \boldsymbol{h}^{(2)} \\ \vdots & \vdots \\ \hat{\boldsymbol{g}}^{(2,r)} & \boldsymbol{h}^{(r)} \end{bmatrix} = \boldsymbol{Q} \begin{bmatrix} \hat{\boldsymbol{f}}^{(2,2)} & \hat{\boldsymbol{h}}^{(2)} \\ \vdots & \vdots \\ \hat{\boldsymbol{f}}^{(2,r)} & \hat{\boldsymbol{h}}^{(r)} \end{bmatrix}$$

In this paper it is assumed that $h_{11} \neq 0$. Comparing the first row and first column elements of the matrices on both sides of Eq. (19), we obtain

$$\lambda^{(2)}_1 = -\hat{f}^{(2)}_{11} / h_{11} \qquad \text{when} \qquad \nu > 2 \tag{20a}$$

$$\lambda^{(2)^2}_1 + h_{11} \lambda^{(2)}_1 + \hat{f}^{(2)}_{11} = 0 \qquad \text{when} \qquad \nu = 2 \tag{20b}$$

When $\nu = 2$, h_{11} and $\hat{f}^{(2)}_{11}$ are only dependent on $\Lambda^{(1)^\nu}$ [that is, for different $\Lambda^{(1)}$ we have same Eq. (20b)]. In this paper it is assumed that when $\nu = 2$ the two roots of Eq. (20b) are distinct. So we have

$$2\lambda_1^{(2)} + h_{11} \neq 0 \tag{21}$$

Thus $\lambda_1^{(2)}$ can be calculated, and $\Lambda^{(2)}$ has been determined completely. Therefore $\boldsymbol{f}^{(2)}$ in Eq. (16) now is known:

$$\boldsymbol{f}^{(2)} = [\nu(\nu-1)/2]\Lambda^{(1)^{\nu-2}}\Lambda^{(2)^2} + \nu\tilde{\boldsymbol{C}}^{(1,1)}\Lambda^{(1)^{\nu-1}}\Lambda^{(2)} + \boldsymbol{h}\Lambda^{(2)} + \hat{\boldsymbol{f}}^{(2)}$$

Thus, $\tilde{\boldsymbol{C}}^{(1,2)}$ and $\lambda_j^{(3)}$ $[j = 2, \ldots, s(1)]$ can be calculated from Eq. (16) by use of the same method in case I, but $\lambda_1^{(3)}$ cannot be determined yet.

Substituting Eq. (18) and all of the preceding results into the equation for $\boldsymbol{W}^{(\nu+3)}$ in Eqs. (5b), we obtain

$$\begin{aligned}\tilde{\boldsymbol{A}}\boldsymbol{W}^{(\nu+3)} &= \sum_{k=1}^{r}\boldsymbol{U}^{(k,\nu)}\Bigg\{\boldsymbol{C}^{(k,3)}\Lambda^{(1)^{\nu}} + \nu\boldsymbol{C}^{(k,2)}\Lambda^{(1)^{\nu-1}}\Lambda^{(2)} \\ &+ \boldsymbol{C}^{(k,1)}\left[\nu\Lambda^{(1)^{\nu-1}}\Lambda^{(3)} + \frac{\nu(\nu-1)}{2}\Lambda^{(1)^{\nu-2}}\Lambda^{(2)^2}\right] \\ &+ \boldsymbol{C}^{(k,0)}\Bigg[\nu\Lambda^{(1)^{\nu-1}}\Lambda^{(4)} + \nu(\nu-1)\Lambda^{(1)^{\nu-2}}\Lambda^{(2)}\Lambda^{(3)} \\ &+ \frac{\nu(\nu-1)(\nu-2)}{6}\Lambda^{(1)^{\nu-3}}\Lambda^{(2)^3}\Bigg]\Bigg\} + \tilde{\boldsymbol{W}}^{(\nu+2)}\Lambda^{(1)} + \tilde{\boldsymbol{W}}^{(\nu+1)}\Lambda^{(2)} \\ &+ \tilde{\boldsymbol{W}}^{(\nu)}\Lambda^{(3)} - \boldsymbol{B}\boldsymbol{W}^{(3)} + \boldsymbol{G}^{(3)} \overset{\text{def}}{=} \boldsymbol{R}^{(3)} + \boldsymbol{G}^{(3)}\end{aligned} \tag{22}$$

All of the terms except $\boldsymbol{R}^{(3)}$ on the right-hand side of Eq. (22) are in $\Re(\tilde{\boldsymbol{A}})$. Define the following known quantities:

$$\boldsymbol{T}^{(1)} = \sum_{k=1}^{r}\boldsymbol{U}^{(k,2)}\left[\boldsymbol{C}^{(k,1)}\Lambda^{(2)} + \boldsymbol{C}^{(k,2)}\Lambda^{(1)}\right]$$

$$\boldsymbol{T}^{(2)} = \boldsymbol{T}^{(1)} + \sum_{k=1}^{r}\boldsymbol{U}^{(k,3)}\left[2\boldsymbol{C}^{(k,0)}\Lambda^{(1)}\Lambda^{(2)} + \boldsymbol{C}^{(k,1)}\Lambda^{(1)^2}\right]$$

$$\hat{\boldsymbol{W}}^{(3)} = \begin{cases}\tilde{\boldsymbol{W}}^{(3)} + \boldsymbol{T}^{(1)}, & \nu = 2\\ \tilde{\boldsymbol{W}}^{(3)} + \boldsymbol{T}^{(2)}, & \nu = 3\\ \boldsymbol{T}^{(2)} + \sum_{k=1}^{r}\boldsymbol{U}^{(k,4)}\boldsymbol{C}^{(k,0)}\Lambda^{(1)^3}, & \nu > 3\end{cases}$$

$$\hat{\boldsymbol{f}}^{(3,j)} = \boldsymbol{V}^{(j)^H}\left[\tilde{\boldsymbol{W}}^{(\nu+2)}\Lambda^{(1)} + \tilde{\boldsymbol{W}}^{(\nu+1)}\Lambda^{(2)} - \boldsymbol{B}\hat{\boldsymbol{W}}^{(3)}\right] \qquad j = 1, \ldots, r$$

From the solvability condition of Eq. (22), $\boldsymbol{V}^{(j)H}\boldsymbol{R}^{(3)} = 0$ $(j = 1, \ldots, r)$, it follows that

$$\begin{aligned}\Lambda^{(1)^\nu}\tilde{\boldsymbol{C}}^{(1,3)} - \tilde{\boldsymbol{C}}^{(1,3)}\Lambda^{(1)^\nu} &= \nu\Lambda^{(1)^{\nu-1}}\Lambda^{(4)} \\ &+ \nu(\nu-1)\Lambda^{(1)^{\nu-2}}\Lambda^{(2)}\Lambda^{(3)} + \boldsymbol{h}\Lambda^{(3)} + \hat{\boldsymbol{f}}^{(3)}\end{aligned} \tag{23}$$

where $\tilde{\boldsymbol{C}}^{(1,3)} = \boldsymbol{C}^{(1,0)^{-1}}\boldsymbol{C}^{(1,3)}$ and the known quantity $\hat{\boldsymbol{f}}^{(3)} = [\hat{f}_{\mathrm{ml}}^{(3)}]$ is defined by

$$\begin{aligned}\hat{\boldsymbol{f}}^{(3)} &= \boldsymbol{C}^{(1,0)^{-1}}\left[\hat{\boldsymbol{f}}^{(3,1)} - \sum_{k=2}^{r}\boldsymbol{Q}^{(1,k)}\hat{\boldsymbol{g}}^{(3,k)}\right] + \frac{\nu(\nu-1)(\nu-2)}{6} \\ &\times \Lambda^{(1)^{\nu-3}}\Lambda^{(2)^3} + \tilde{\boldsymbol{C}}^{(1,1)}\left[\nu\Lambda^{(1)^{\nu-1}}\Lambda^{(3)} + \frac{\nu(\nu-1)}{2}\Lambda^{(1)^{\nu-2}}\Lambda^{(2)^2}\right] \\ &+ \nu\tilde{\boldsymbol{C}}^{(1,2)}\Lambda^{(1)^{\nu-1}}\Lambda^{(2)}\end{aligned}$$

In the preceding equations the known quantities $\hat{\boldsymbol{g}}^{(3,k)}$ $(k = 2, \ldots, r)$ are defined by

$$\begin{bmatrix}\hat{\boldsymbol{g}}^{(3,2)}\\ \vdots\\ \hat{\boldsymbol{g}}^{(3,r)}\end{bmatrix} = \boldsymbol{Q}\begin{bmatrix}\hat{\boldsymbol{f}}^{(3,2)}\\ \vdots\\ \hat{\boldsymbol{f}}^{(3,r)}\end{bmatrix}$$

Although $\lambda_1^{(3)}$ is not known now, some matrices such as $\Lambda^{(1)^{\nu-1}}\Lambda^{(3)}$, which contain the product of positive powers of $\Lambda^{(1)}$ and $\Lambda^{(3)}$, are known because $\lambda_1^{(1)} = 0$. Comparing the first row and first column elements of the matrices on both sides of Eq. (23), we obtain the first diagonal element of $\Lambda^{(3)}$:

$$\lambda_1^{(3)} = \begin{cases}-\hat{f}_{11}^{(3)}/h_{11}, & \nu > 2\\ -\hat{f}_{11}^{(3)}/\left[2\lambda_1^{(2)} + h_{11}\right], & \nu = 2\end{cases} \tag{24}$$

The condition (21) has been used in the second expression on the right-hand side of Eq. (24). Thus, $\Lambda^{(3)}$ has been determined completely, and

$$\boldsymbol{f}^{(3)} = \nu(\nu-1)\Lambda^{(1)^{\nu-2}}\Lambda^{(2)}\Lambda^{(3)} + \boldsymbol{h}\Lambda^{(3)} + \hat{\boldsymbol{f}}^{(3)}$$

is known. By use of the similar way, $\tilde{\boldsymbol{C}}^{(1,3)}$ and $\lambda_j^{(4)}[j = 2, \ldots, s(1)]$ can be calculated from Eq. (23), but $\lambda_1^{(4)}$ cannot be determined up to now. To calculate $\lambda_1^{(4)}$, we have to use the solvability condition of the equation for $\boldsymbol{W}^{(\nu+4)}$. Usually in order to calculate the first diagonal element of $\Lambda^{(k)}$, we have to use the perturbation equation that is one order higher than the equation needed in determining the other diagonal elements of $\Lambda^{(k)}$.

Numerical Example

Consider the 12×12 matrix $\boldsymbol{A}$ in Ref. 6 and the 12×12 matrix

$$\boldsymbol{B} = \begin{bmatrix}3 & -4 & 3 & 0 & -2 & 0 & 2 & -1 & 1 & -2 & 1 & 0\\ 4 & -7 & 6 & 0 & -4 & 0 & 4 & -2 & 2 & -4 & 2 & 0\\ 4 & -8 & 8 & 0 & -6 & 0 & 6 & -3 & 3 & -6 & 3 & 0\\ 4 & -9 & 10 & 1 & -8 & 0 & 8 & -4 & 4 & -8 & 4 & 0\\ 4 & -9 & 10 & 2 & -10 & 0 & 10 & -5 & 5 & -10 & 5 & 0\\ 4 & -9 & 10 & 2 & -12 & 0 & 12 & -6 & 6 & -12 & 6 & 0\\ 4 & -9 & 10 & 2 & -12 & -1 & 14 & -7 & 7 & -14 & 7 & 0\\ 4 & -9 & 10 & 2 & -12 & -2 & 16 & -8 & 8 & -16 & 8 & 0\\ 4 & -9 & 10 & 2 & -12 & -3 & 18 & -9 & 9 & -18 & 9 & 0\\ 4 & -9 & 10 & 2 & -12 & -3 & 18 & -9 & 10 & -20 & 10 & 0\\ 4 & -9 & 10 & 2 & -12 & -3 & 18 & -9 & 11 & -22 & 11 & 0\\ 4 & -9 & 10 & 2 & -12 & -3 & 18 & -9 & 12 & -24 & 12 & 0\end{bmatrix}$$

All of the eigenvalues of $\boldsymbol{A}$ are 1. In the Jordan canonical form of $\boldsymbol{A}$, there are three blocks of order 2 and two blocks of order 3. We can take $U^{(1,1)}$, $U^{(1,2)}$, $U^{(2,1)}$, $U^{(2,2)}$, $U^{(2,3)}$, $V^{(1)}$, $V^{(2)}$ as in Ref. 6. For sufficiently small $\varepsilon > 0$ the eigenvalues of $\boldsymbol{A} + \varepsilon\boldsymbol{B}$ are $1 + \eta^2$, $1 + \eta$, $1 + \eta i$; $1 - \eta^2$, $1 - \eta$, $1 - \eta i$; $1 + \eta$, $1 - \eta$; $1 - \eta\omega_-$, $1 + \eta\omega_+$; $1 - \eta\omega_+$, $1 + \eta\omega_-$; where $\omega_\pm = \frac{1}{2} \pm \sqrt{3}i/2$ and $\eta = \varepsilon^{1/2}$ for the first two groups of the eigenvalues and $\eta = \varepsilon^{1/3}$ for the last three groups of the eigenvalues. The first- to third-order perturbation coefficients of the eigenvalues in the preceding five groups are, respectively, (0, 1, i; 1, 0, 0; 0, 0, 0), (0, −1, $-i$; −1, 0, 0; 0, 0, 0), (1, −1; 0, 0; 0, 0), ($-\omega_-$, ω_+; 0, 0; 0, 0), ($-\omega_+$, ω_-; 0, 0; 0, 0). If associated with the jth group of the eigenvalues, we denote the matrix of the eigenvectors of the perturbed problem, that of the differentiable eigenvectors, and that of the kth-order perturbation coefficients of the eigenvectors of the unperturbed problem by $\boldsymbol{W}_j(\eta)$, $\boldsymbol{W}_j^{(0)}$, and $\boldsymbol{W}_j^{(k)}$, respectively ($j = 1, \ldots, 5$; $k = 1, 2, \ldots$), then we have

$$
\boldsymbol{W}_1(\eta) = \begin{bmatrix} 1 & \frac{1}{3} & 0.2 \\ 1 & \frac{2}{3} & 0.4 \\ 1 & 1 & 0.6 \\ 1 & \rho_1 & 0.8 \\ 1 & \rho_1 & 1 \\ 1 & \rho_1 & \rho_2 \\ \vdots & \vdots & \vdots \\ 1 & \rho_1 & \rho_2 \end{bmatrix}, \qquad \boldsymbol{W}_2(\eta) = \begin{bmatrix} 1 & \frac{1}{3} & 0.2 \\ \rho_3 & \frac{2}{3} & 0.4 \\ \rho_3 & 1 & 0.6 \\ \rho_3 & \rho_4 & 0.8 \\ \rho_3 & \rho_4 & 1 \\ \rho_3 & \rho_4 & \overline{\rho_2} \\ \vdots & \vdots & \vdots \\ \rho_3 & \rho_4 & \overline{\rho_2} \end{bmatrix}
$$

$$
\boldsymbol{W}_3(\eta) = \begin{bmatrix} \frac{1}{7} & 0.1 \\ \frac{2}{7} & 0.2 \\ \vdots & \vdots \\ \frac{6}{7} & 0.6 \\ 1 & 0.7 \\ \rho_5 & 0.8 \\ \rho_6 & 0.9 \\ \rho_6 & 1 \\ \rho_6 & \rho_7 \\ \rho_6 & \rho_8 \end{bmatrix}, \qquad \boldsymbol{W}_4(\eta) = \begin{bmatrix} \frac{1}{7} & 0.1 \\ \frac{2}{7} & 0.2 \\ \vdots & \vdots \\ \frac{6}{7} & 0.6 \\ 1 & 0.7 \\ \rho_9 & 0.8 \\ \rho_{10} & 0.9 \\ \rho_{10} & 1 \\ \rho_{10} & \rho_{11} \\ \rho_{10} & \rho_{12} \end{bmatrix}
$$

$$
\boldsymbol{W}_5(\eta) = \overline{\boldsymbol{W}_4(\eta)}
$$

where

$$
\rho_1 = \frac{3+4\eta}{3+3\eta}, \qquad \rho_2 = \frac{5+6i\eta}{5+5i\eta}, \qquad \rho_3 = \frac{1-4\eta^2}{1-2\eta^2}
$$

$$
\rho_4 = \frac{3-4\eta}{3-3\eta}, \qquad \rho_5 = \frac{7+8\eta+8\eta^2}{\xi_1}, \qquad \rho_6 = \frac{7+8\eta+9\eta^2}{\xi_1}
$$

$$
\rho_7 = \frac{10-11\eta+11\eta^2}{\xi_2}, \qquad \rho_8 = \frac{10-11\eta+12\eta^2}{\xi_2}
$$

$$
\rho_9 = \frac{7-8\eta\omega_- - 8\eta^2\omega_+}{\xi_3}, \qquad \rho_{10} = \frac{7-8\eta\omega_- - 9\eta^2\omega_+}{\xi_3}
$$

$$
\rho_{11} = \frac{10+11\eta\omega_+ - 11\eta^2\omega_-}{\xi_4}, \qquad \rho_{12} = \frac{10+11\eta\omega_+ - 12\eta^2\omega_-}{\xi_4}
$$

and $\xi_1 = 7(1+\eta+\eta^2)$, $\xi_2 = 10(1-\eta+\eta^2)$, $\xi_3 = 7(1-\eta\omega_- - \eta^2\omega_+)$, $\xi_4 = 10(1+\eta\omega_+ - \eta^2\omega_-)$. In the preceding expressions, as defined before, $\eta = \varepsilon^{1/2}$ for the first two groups and $\eta = \varepsilon^{1/3}$ for the last three groups.

The calculations were conducted with 16 significant decimal digits. The generalized{1}-inverses of singular matrices were computed by Gaussian elimination with complete pivot. All of the calculated perturbation coefficients of the eigenvalues are correct almost within the machinery accuracy. With at least 13 significant decimal digits the calculated differentiable eigenvectors and the perturbation coefficients of the eigenvectors are

$$
\boldsymbol{W}_1^{(0)} = \boldsymbol{W}_2^{(0)} = \begin{bmatrix} 1 & \frac{1}{3} & 0.2 \\ 1 & \frac{2}{3} & 0.4 \\ 1 & 1 & 0.6 \\ 1 & 1 & 0.8 \\ 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \end{bmatrix}, \qquad \boldsymbol{W}_1^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0.2i \\ \vdots & \vdots & \vdots \\ 0 & \frac{1}{3} & 0.2i \end{bmatrix}
$$

$$
\boldsymbol{W}_1^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0.2 \\ \vdots & \vdots & \vdots \\ 0 & -\frac{1}{3} & 0.2 \end{bmatrix}, \qquad \boldsymbol{W}_2^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & -0.2i \\ \vdots & \vdots & \vdots \\ 0 & -\frac{1}{3} & -0.2i \end{bmatrix}
$$

$$
\boldsymbol{W}_2^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \\ -2 & -\frac{1}{3} & 0 \\ -2 & -\frac{1}{3} & 0 \\ -2 & -\frac{1}{3} & 0.2 \\ \vdots & \vdots & \vdots \\ -2 & -\frac{1}{3} & 0.2 \end{bmatrix}, \qquad \boldsymbol{W}_3^{(0)} = \boldsymbol{W}_4^{(0)} = \boldsymbol{W}_5^{(0)} = \begin{bmatrix} \frac{1}{7} & 0.1 \\ \frac{2}{7} & 0.2 \\ \vdots & \vdots \\ \frac{6}{7} & 0.6 \\ 1 & 0.7 \\ 1 & 0.8 \\ 1 & 0.9 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}
$$

$$
\boldsymbol{W}_3^{(1)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \frac{1}{7} & 0 \\ \frac{1}{7} & 0 \\ \frac{1}{7} & 0 \\ \frac{1}{7} & -0.1 \\ \frac{1}{7} & -0.1 \end{bmatrix}, \qquad \boldsymbol{W}_3^{(2)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{7} & 0 \\ \frac{1}{7} & 0 \\ \frac{1}{7} & 0 \\ \frac{1}{7} & -0.1 \end{bmatrix}
$$

$$
\boldsymbol{W}_4^{(1)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ -\omega_-/7 & 0 \\ -\omega_-/7 & 0 \\ -\omega_-/7 & 0 \\ -\omega_-/7 & 0.1\omega_+ \\ -\omega_-/7 & 0.1\omega_+ \end{bmatrix}, \qquad \boldsymbol{W}_4^{(2)} = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \\ -\omega_+/7 & 0 \\ -\omega_+/7 & 0 \\ -\omega_+/7 & 0 \\ -\omega_+/7 & -0.1\omega_- \end{bmatrix}
$$

$$
\boldsymbol{W}_5^{(1)} = \overline{\boldsymbol{W}_4^{(1)}}, \qquad \boldsymbol{W}_5^{(2)} = \overline{\boldsymbol{W}_4^{(2)}}
$$

By direct Taylor expansion of $\boldsymbol{W}_j(\eta)$ respect to η $(j = 1, \ldots, 5)$, we can see that the preceding calculated results are correct.

Conclusions

Under the assumption that Eqs. (2) hold, which means that all of the first-order perturbation coefficients of the eigenvalues are nonzeros, and the assumption that all of the eigenvalues of problem (7) are simple, this paper gives a direct method to calculate the first- to third-order perturbation coefficients of the eigenvalues and the first- to second-order perturbation coefficients of the eigenvectors of a defective matrix. The method is extended to the case where $\Delta_1 = 0$ and $\Delta_k \neq 0$ $(k = 2, \ldots, r)$, which means that the first-order perturbation coefficients of eigenvalues associated with the lowest-order Jordan blocks can have zero value. Numerical examples show the validity of the method. If we have $\Delta_k = 0$ for some $k > 1$, then the earlier methods cannot be used, and in this case we have to find some new way to do higher-order eigensensitivity analysis.

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